

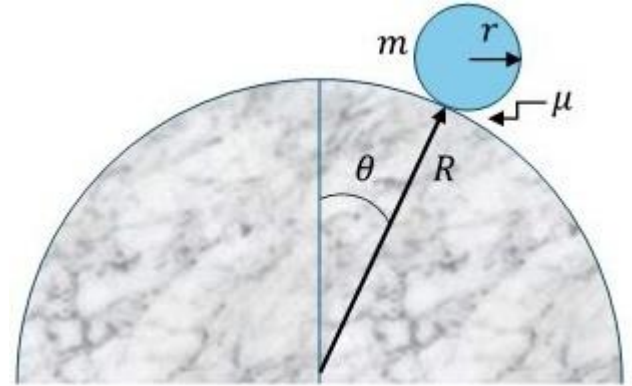


PHYS 101 – General Physics I Final Exam

Duration: 120 minutes

Sunday, 18 May 2025; 09:30

1. A cylinder of mass m and radius r is on top of a larger immobile half cylinder of radius R . There is enough friction between the cylinders so that the upper cylinder can always roll without slipping. The upper cylinder, which has a moment of inertia $mr^2/2$ around its symmetry axis, is released from the top point ($\theta = 0$) with a negligible initial velocity, and it rolls down and loses contact with the lower cylinder. Gravitational acceleration is $\vec{g}$.



(a) (12 Pts.) Find the center of mass velocity of the cylinder after it has rolled by an angle θ , as shown in the figure.

(b) (13 Pts.) At which angle θ do the two cylinders lose contact?

Solution:

(a) Total mechanical energy is conserved. Let $U_g = 0$ when $\theta = 0$.

$$E_i = 0, \quad E(\theta) = \frac{1}{2}mv_c^2 + \frac{1}{2}I\omega^2 - mg(R+r)(1 - \cos \theta)$$

Since the cylinder is rolling, the velocity of its center satisfies $v_c = r\omega$. We have

$$\frac{1}{2}mv_c^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 - mg(R+r)(1 - \cos \theta) = 0 \rightarrow v_c = \sqrt{\frac{4}{3}g(R+r)(1 - \cos \theta)}$$

(b) The free body diagram of the cylinder implies

$$mg \cos \theta - n = m \frac{v_c^2}{R+r}$$

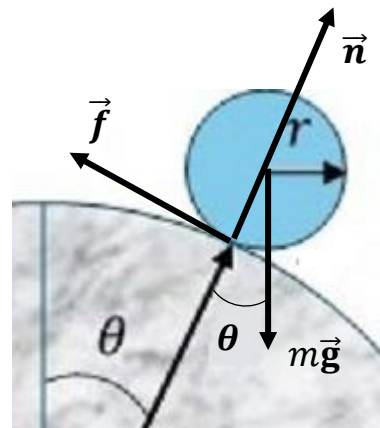
The cylinder loses contact when $n = 0$, Therefore

$$mg \cos \theta = m \frac{v_c^2}{R+r}.$$

Using the expression for v_c found in part (a), we get

$$mg \cos \theta = \frac{4}{3}mg(1 - \cos \theta) \rightarrow \cos \theta = \frac{4}{7}.$$

$$\theta = \arccos\left(\frac{4}{7}\right)$$



2. A block of mass m is moving with speed v_0 on a horizontal surface towards two stationary blocks with the same mass, which are connected by a spring of stiffness constant k . Assume that the collision is instantaneous, and that there is no friction in the system.

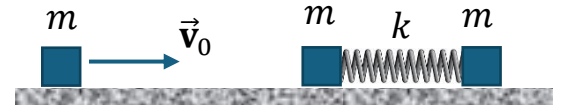
If the collision is elastic:

(a) (10 Pts.) What will be maximum compression of the spring in the resulting motion?

If the collision is completely inelastic:

(b) (5 Pts.) What will be the center of mass speed of the whole system?

(c) (10 Pts.) What will be the oscillation frequency of the spring-mass system?



Solution: (a) If the collision is elastic, both linear momentum and kinetic energy will be conserved in the collision.

$$mv_0 = mv'_1 + mv'_2 \rightarrow v'_1 + v'_2 = v_0, \quad \frac{1}{2}mv_0^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 \rightarrow -v'_1 + v'_2 = v_0$$

Solving the two equations, we get $v'_1 = 0$, $v'_2 = v_0$. So, following the collision, the block which is initially moving will stop, and the target block will start moving with speed v_0 . The velocity of the center of mass and the total mechanical energy of the two block system after the collision are

$$v_{cm} = v_0/2, \quad E = \frac{1}{2}(2m)v_{cm}^2 + E_{rel}$$

where E_{rel} is the energy of the two blocks relative to the center of mass. At maximum compression of the spring, the two blocks will be at rest relative to each other therefore all relative energy will be in potential energy form. Since total mechanical energy is conserved, we have

$$\frac{1}{4}mv_0^2 + \frac{1}{2}k\Delta^2 = \frac{1}{2}mv_0^2 \rightarrow \Delta = \sqrt{\frac{mv_0^2}{2k}} = v_0\sqrt{\frac{m}{2k}}.$$

(b) Since linear momentum is conserved,

$$3m v_{cm} = m v_0 \rightarrow v_{cm} = \frac{v_0}{3}.$$

(c) The system will be oscillating around the center of mass. If we let x_1 and x_2 denote displacements relative to the center of mass of the two masses from equilibrium length of the spring, we have $2mx_1 - mx_2 = 0$. Total extension of the spring is $x_1 + x_2 = 3x_1$. Therefore, the force acting on the $2m$ mass will be $F = -3kx_1$. Newton's second law is written as

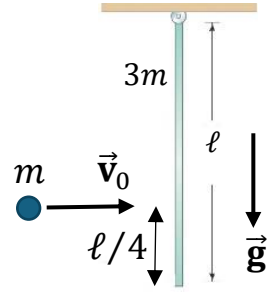
$$2m \frac{d^2x_1}{dt^2} = -3kx_1 \rightarrow \frac{d^2x_1}{dt^2} = -\left(\frac{3k}{2m}\right)x_1 \rightarrow \omega = \sqrt{\frac{3k}{2m}}.$$

3. A uniform rod of mass $3m$ and length ℓ is connected to the ceiling by a frictionless pivot, and is at rest in a vertical position. The moment of inertia of the rod around its center of mass is $m\ell^2/4$. A small particle of mass m collides with the rod with a horizontal velocity $\vec{v}_0$ at $\ell/4$ above the lowest point of the rod. Gravitational acceleration is g .

If the collision is elastic:

(a) (6 Pts.) What is the rebound velocity of the particle? (Assume that it rebounds with a horizontal velocity)

(b) (6 Pts.) How high would the rod swing? (Give the cosine of the largest angle it makes with the vertical)



If the collision is completely inelastic

(c) (13 Pts.) What is the frequency of small oscillation of the combined rod-particle system?

Solution: Angular momentum with respect to the pivot is conserved during the collision. Let v' be the velocity of the particle and let ω be the angular speed of the rod immediately after the collision. The moment of inertia of the rod about the pivot is found by using the parallel axis theorem as $I_p = I_{cm} + 3m\left(\frac{\ell}{2}\right)^2 = m\ell^2$. Hence

$$L_i = mv_0\left(\frac{3\ell}{4}\right) = \frac{3}{4}mv_0\ell, \quad L_f = mv'\left(\frac{3\ell}{4}\right) + I_p\omega = \frac{3}{4}mv'\ell + I_p\omega, \quad L_i = L_f \rightarrow v' = v_0 - \frac{4}{3}\ell\omega$$

(a) If the collision is elastic, kinetic energy is also conserved in the collision. So

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv'^2 + \frac{1}{2}(m\ell^2)\omega^2 \rightarrow v_0^2 = v'^2 + \ell^2\omega^2 \rightarrow v_0^2 = \left(v_0 - \frac{4}{3}\ell\omega\right)^2 + \ell^2\omega^2$$

The rebound velocity of the particle is found as $v' = -7v_0/25$, and $\omega = 24v_0/25\ell$.

(b) Kinetic energy of the rod immediately after the collision is $K = I\omega^2/2$. At the highest point of the swing, this will be equal to the gravitational potential energy of the center of mass of the rod. Therefore

$$\frac{1}{2}I\omega^2 = 3mg\left(\frac{\ell}{2}\right)(1 - \cos\theta_m) \rightarrow \cos\theta_m = 1 - \frac{1}{3g\ell}\left(\frac{24v_0}{25}\right)^2.$$

(c) If the collision is completely inelastic, the particle will stick to the rod. The center of mass of the system will be at a distance $r_{cm} = 9\ell/16$ from the pivot. The restoring torque will be $\tau = -4mg(9\ell/16)\sin\theta$. Total moment of inertia of the rod plus the particle will be $I = (25/16)m\ell^2$. Therefore, Newton's second law is written as

$$I\frac{d^2\theta}{dt^2} = \tau \rightarrow \frac{25}{16}m\ell^2\frac{d^2\theta}{dt^2} = -\frac{9\ell}{4}mg\sin\theta \rightarrow \frac{d^2\theta}{dt^2} = -\left(\frac{36g}{25\ell}\right)\sin\theta.$$

for small oscillations $\sin\theta \approx \theta$, so

$$\frac{d^2\theta}{dt^2} = -\left(\frac{36g}{25\ell}\right)\theta \rightarrow \omega = \frac{6}{5}\sqrt{\frac{g}{\ell}}.$$

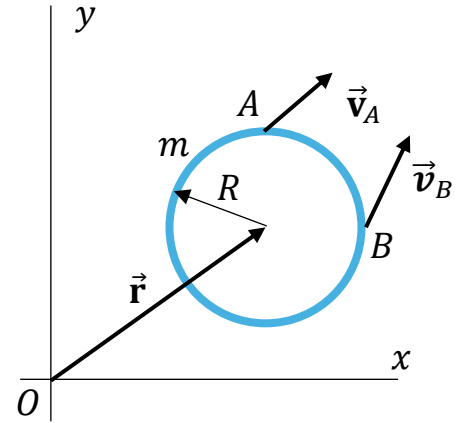
4. A thin ring is rotating and sliding without friction on ice. The radius of the ring is R , and its mass is m . At a particular moment in time the center of the ring is at position $\vec{r} = x_0 \hat{i} + y_0 \hat{j}$. At this moment, the velocity of the topmost point of the ring (point A) with respect to the origin is $\vec{v}_A = v_0 \hat{i} + 2v_0 \hat{j}$, while the x component of the velocity of the rightmost point of the ring (point B) with respect to the origin is $v_{Bx} = 2v_0$.

(a) (8 Pts.) What is the angular velocity of rotation of the ring with respect to its center of mass? Is it clockwise or counter clockwise? (Explain.)

(b) (7 Pts.) What is the velocity of the center of mass of the ring with respect to the origin O ?

(c) (5 Pts.) What is the kinetic energy of the ring?

(d) (5 Pts.) What is the magnitude of the angular momentum of the ring with respect to the origin?



Solution: (a) Given that $\vec{v}_A = v_0 \hat{i} + 2v_0 \hat{j}$, and $\vec{v}_B = 2v_0 \hat{i} + v_y \hat{j}$.

Let $\vec{r}'_A, \vec{v}'_A$ and $\vec{r}'_B, \vec{v}'_B$ denote positions and velocities of the points A and B respectively, relative to the center of the ring, and $\vec{v}_C$ denote the velocity of the center of the ring. Hence, $\vec{v}_A = \vec{v}_C + \vec{v}'_A$, and $\vec{v}_B = \vec{v}_C + \vec{v}'_B$, which means $\vec{v}_A - \vec{v}_B = \vec{v}'_A - \vec{v}'_B$. Since the ring is rotating rigidly, $\vec{v}'_A$ is perpendicular to $\vec{r}'_A$, and $\vec{v}'_B$ is perpendicular to $\vec{r}'_B$, therefore $\vec{v}'_A - \vec{v}'_B$ must be perpendicular to $\vec{r}'_A - \vec{r}'_B$, or

$$[v_0 \hat{i} + 2v_0 \hat{j} - (2v_0 \hat{i} + v_y \hat{j})] \cdot (\hat{i} - \hat{j}) = 0 \rightarrow v_y = 3v_0 \rightarrow \vec{v}_A - \vec{v}_B = -v_0(\hat{i} + \hat{j}).$$

Assuming that the direction of rotation is counter clockwise, we have

$$\vec{v}_A = \vec{v}_C - R\omega \hat{i}, \text{ and } \vec{v}_B = \vec{v}_C + R\omega \hat{j} \rightarrow \vec{v}_A - \vec{v}_B = -R\omega(\hat{i} + \hat{j}) = -v_0(\hat{i} + \hat{j}) \rightarrow \omega = \frac{v_0}{R}.$$

Since $\omega > 0$, our assumption of counter clockwise rotation is true.

(b) Using above results,

$$\vec{v}_C = \vec{v}_A + R\omega \hat{i} \rightarrow \vec{v}_C = 2v_0 \hat{i} + 2v_0 \hat{j}$$

(c)

$$K = \frac{1}{2}mv_C^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m(8v_0^2) + \frac{1}{2}(mR^2)\left(\frac{v_0}{R}\right)^2 = \frac{9}{2}mv_0^2$$

(d)

$$\vec{L} = \vec{r} \times m\vec{v}_C + I\vec{\omega} \rightarrow \vec{L} = [2m(x_0 - y_0)v_0 + mRv_0] \hat{k}$$

$$|\vec{L}| = [2(x_0 - y_0) + R]mv_0$$